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Transonic Small Disturbance Potential Equation

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An analytical solution is presented to the transonic small disturbance potential equation with the Kutta–Joukowsky boundary conditions for an airfoil of zero thickness, generalizing it at the same time to nonzero angle of attack. In particular, an extension of the Possio integral equation is derived that is valid for nonzero angle of attack, as well as a formula for the divergence speed, showing explicitly a transonic dip, depending on the angle of attack.

I. Introduction

THE three-dimensional transonic small disturbance potential (TSD) equation is given by

$$\frac{\partial^2\phi}{\partial t^2} + 2U\frac{\partial^2\phi}{\partial t\partial x} = a_\infty^2\left((1-M^2)-(1+\gamma)M^2\frac{\partial\phi}{\partial x}\right)\frac{\partial^2\phi}{\partial x^2} + a_\infty^2\frac{\partial^2\phi}{\partial y^2} + a_\infty^2\frac{\partial^2\phi}{\partial z^2}$$

$$-\infty < x, y, z < \infty, \qquad 0 \le t < \infty, \qquad U = Ma_\infty \tag{1}$$

with the boundary conditions to be specified later. As Nixon notes in his review paper,[1] "... in the author's opinion the minimum complexity of equation that should be used for transonic flow prediction is (i) since equations lower in the hierarchy, e.g., the linear subsonic equation cannot represent shock waves."

The purpose of this paper is to present an analytical (as opposed to numerical computation) solution of this equation, generalizing at the same time to nonzero angle of attack. This solution is different in nature from the small parameter expansion suggested by Nixon.[1]

We begin in Sec. II with a derivation of this equation, where we use a different path from Nixon's[1] at a crucial point and correct an error therein at the same time. The extension of the Possio integral equation for nonzero angle of attack and the new analytical solution, a Volterra expansion, to the boundary-input problem for the TSD equation is in Sec. III. It uses crucially the solution to the linearized TSD equation, which in turn is shown to yield a new extension of the Possio integral equation to allow nonzero angle of attack. In Sec. IV, Discussion, we derive, as a direct byproduct of the theory, a new formula for the divergence speed for a slender (zero thickness) straight high-aspect-ratio wing, indicating the effect of nonzero angle of attack. It is remarkable that it already exhibits a transonic dip, making it plausible that this should also be reflected in the flutter boundary. Conclusions are in the final section, Sec. V.

II. Full Potential Equation

We start with a derivation of the full potential equation first from the basic equations of fluid dynamics, drawing on Nixon,[1] Myers,[2] Chorin and Marsden,[3] and Cole and Cook,[4] and clarify assumptions made and some obscure points.

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The continuity equation is

$$\dot\rho + \nabla\cdot(\rho\boldsymbol{q}) = 0 \tag{2}$$

where ρ is the density, $\rho(t, x, y, z)$, and for simplicity of notation we inhibit the independent variables, following custom, $\boldsymbol{q}$ is the vector velocity, and the superdot represents the time derivative.

Euler Equation

Because we consider only inviscid flow, we may replace the Navier–Stokes equation by the momentum-balance Euler equation

$$\frac{\mathrm{D}\boldsymbol{q}}{\mathrm{D}t} + \frac{\nabla p}{\rho} = 0 \tag{3}$$

where $\mathrm{D}/\mathrm{D}t$ is the total derivative with respect to t and p is the pressure.

Energy (Thermodynamics) Equations

As is well known,[1–4] we need to include the thermodynamic relations to assure uniqueness of solution. Let S denote the entropy and T the temperature. Together with p and ρ, we have four variables of which only two are independent.[2] Let h denote the enthalpy given by

$$h = c_pT \tag{4}$$

where c_p is the specific heat at constant pressure. Then we have to satisfy the Gibbs relation (which we state in the usual form):

$$T\,\mathrm{d}S + \mathrm{d}p/\rho = \mathrm{d}h \tag{5}$$

Next we make the key assumption: The isentropy condition

$$\mathrm{d}S = 0 \tag{6}$$

and perfect gas (with no conduction), so that we have the gas law

$$p = \rho RT \tag{7}$$

where

$$R = c_p - c_v$$

where c_v is the specific heat at constant volume. Then we can deduce a fundamental relation, which we state as a theorem.

Theorem 1. From Eqs. (5–7), we have

$$p = \left(p_\infty/\rho_\infty^\gamma\right)\rho^\gamma \tag{8}$$

where

$$\gamma = c_p/c_v \ge 1$$

and p_∞ and ρ_∞ are freestream values of p and ρ, respectively, as $|x|, |y|, |z| \to \infty$, and where

$$p_\infty/\rho_\infty = a_\infty^2$$

where a_∞ is the freestream or far-field speed of sound in the medium.

Proof. From Eqs. (4) and (7), we have

$$h = [c_p/(c_p - c_v)](p/\rho) = [\gamma/(\gamma - 1)](p/\rho) \tag{9}$$

where γ is given by Eq. (8) and $\gamma \geq 1$ so that h is nonnegative and so, of course, are p, ρ, and T. By the isentropy condition (6), we have

$$\nabla p/\rho = \nabla h \tag{10}$$

and substituting from Eq. (9),

$$\nabla p/\rho = [\gamma/(\gamma - 1)](\nabla p/\rho) - [\gamma/(\gamma - 1)](p/\rho^2)\nabla\rho$$

Hence, collecting terms, we obtain

$$[\gamma/(\gamma - 1)](p\nabla\rho/\rho^2) = [1/(\gamma - 1)](\nabla p/\rho)$$

or

$$\gamma(\nabla\rho/\rho) = \nabla p/p$$

or

$$\nabla \log(p/\rho^\gamma) = 0$$

Hence,

$$p = \left(p_\infty \big/ \rho_\infty^\gamma\right)\rho^\gamma$$

which is Eq. (8), as required.

In particular, we see that

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \gamma\frac{p_\infty}{\rho_\infty^\gamma}\rho^{\gamma-1} = \gamma\frac{p}{\rho}$$

which by definition[2] is equal to a^2, where a is the local speed of sound. Hence, we also have

$$\gamma(p_\infty/\rho_\infty) = a_\infty^2 \tag{11}$$

Next following Ref. 1, we recall Crocco's equation,

$$T\nabla S + q \times (\nabla \times q) = \nabla h + \frac{\mathrm{D}q}{\mathrm{D}t} \tag{12}$$

Our next result also may be stated as a theorem in view of its importance.

Theorem 2. Under Eqs. (5–7) and now Eq. (12), we may seek a solution in the form

$$\boldsymbol{q} = \nabla\phi$$

where ϕ is the velocity potential, that is, we may consider the flow to be irrotational.

Proof. Under the isotropy conditions, Crocco's equation (12) becomes

$$q \times (\nabla \times q) = \nabla h + \frac{\mathrm{D}q}{\mathrm{D}t} \tag{13}$$

and further by Eq. (10),

$$\nabla h = \nabla p/\rho$$

Hence, the right-hand side of Eq. (13)

$$\nabla h + \frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\nabla p}{\rho} + \frac{\mathrm{D}q}{\mathrm{D}t} = 0$$

by Eq. (3). Hence,

$$\boldsymbol{q} \times (\nabla \times \boldsymbol{q}) = 0 \tag{14}$$

Now the freestream velocity $\boldsymbol{q}_\infty$ is given by

$$\boldsymbol{q}_\infty = \nabla\phi_\infty$$

where we assume

$$\phi_\infty = (x\cos\alpha\cos\beta + y\sin\alpha\cos\beta + z\sin\beta)U, \qquad U > 0 \tag{15}$$

and, hence,

$$\nabla \times \boldsymbol{q}_\infty = 0$$

Hence, Eq. (14) can be satisfied if we can find $\boldsymbol{q}$ such that

$$\nabla \times \boldsymbol{q} = 0$$

the flow being, thus, irrotational, and so can be expressed in the form

$$\boldsymbol{q} = \nabla\phi \tag{16}$$

where ϕ is then the velocity potential, with ϕ_∞ the freestream potential given by Eq. (15). QED

Hence substituting Eq. (16) into the Bernoulli equation

$$\frac{\mathrm{D}q}{\mathrm{D}t} + \nabla h = 0$$

we have

$$\nabla\left[\frac{\mathrm{D}\phi}{\mathrm{D}t} + h\right] = 0$$

and because

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2$$

we obtain

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + h = \frac{1}{2}|q_\infty|^2 + \frac{\gamma}{\gamma - 1}\frac{p_\infty}{\rho_\infty} \tag{17}$$

which is recognized as another version of Bernoulli's equation. To proceed further, we need to express h in terms of ρ. Thus, we have

$$h = [\gamma/(\gamma - 1)](p/\rho) = [\gamma/(\gamma - 1)]\left(p_\infty \big/ \rho_\infty^\gamma\right)\rho^{\gamma-1}$$

$$= [\gamma/(\gamma - 1)](\rho/\rho_\infty)^{\gamma-1}a_\infty^2$$

using Eq. (9). Hence,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{a_\infty^2}{\gamma - 1}\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} = \frac{1}{2}|q_\infty|^2 + \frac{a_\infty^2}{\gamma - 1}$$

or

$$\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} = \frac{\gamma - 1}{a_\infty^2}\left[\frac{1}{2}|q_\infty|^2 + \frac{a_\infty^2}{\gamma - 1} - \frac{\partial\phi}{\partial t} - \frac{1}{2}|\nabla\phi|^2\right]$$

$$= 1 + \frac{\gamma - 1}{a_\infty^2}\left[\frac{1}{2}|q_\infty|^2 - \frac{1}{2}|\nabla\phi|^2 - \frac{\partial\phi}{\partial t}\right] \tag{18}$$

or

$$\frac{\rho}{\rho_\infty} = \left[1 + \frac{\gamma - 1}{a_\infty^2}\left(\frac{1}{2}|q_\infty|^2 - \frac{1}{2}|\nabla\phi|^2 - \frac{\partial\phi}{\partial t}\right)\right]^{1/(\gamma-1)}$$

Nixon[1] has an arithmetical error in formula (20) in his paper, which seems to propagate through the rest of his derivation. Moreover, at this point he proceeds to take logarithms (as indeed Cole and Cook[4] do) and makes approximations. We shall show here that it is not necessary to make any such approximations in the derivation. Thus, from Eq. (18) by differentiating with respect to t [getting ready to use the continuity equation (2)] we have

$$\frac{\partial}{\partial t}\rho^{\gamma-1} = (\gamma - 1)\rho^{\gamma-2}\frac{\partial\rho}{\partial t}$$

$$= \rho_\infty^{\gamma-1}\frac{\gamma - 1}{a_\infty^2}\left(-\frac{\partial^2\phi}{\partial t^2} - \frac{\partial}{\partial t}\frac{1}{2}|\nabla\phi|^2\right)$$

or

$$a_\infty^2 \rho^{\gamma-2} \frac{\partial \rho}{\partial t} = \rho_\infty^{\gamma-1}\left(-\frac{\partial^2\phi}{\partial t^2} - \frac{\partial}{\partial t}\frac{1}{2}|\nabla\phi|^2\right) \tag{19}$$

However, by the continuity equation (2),

$$\rho^{\gamma-2}\frac{\partial\rho}{\partial t} = \rho^{\gamma-2}[-\rho\nabla^2\phi - \nabla\phi\cdot\nabla\rho]$$

$$= -\rho^{\gamma-1}\nabla^2\phi - \frac{\nabla\phi\cdot\nabla\rho^{\gamma-1}}{\gamma-1} \tag{20}$$

Hence, equating Eqs. (20) and (19), we obtain

$$\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\frac{1}{2}|\nabla\phi|^2 = \left[\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}\nabla^2\phi + \frac{\nabla\phi\cdot\nabla(\rho/\rho_\infty)^{\gamma-1}}{\gamma-1}\right]a_\infty^2$$

$$= a_\infty^2\nabla^2\phi\left[1 + \frac{\gamma-1}{a_\infty^2}\left(\frac{|q_\infty|^2}{2} - \frac{\partial\phi}{\partial t} - \frac{\nabla|\phi|^2}{2}\right)\right]$$

$$+ \frac{a_\infty^2}{\gamma-1}\nabla\phi\cdot\left[\frac{\gamma-1}{a_\infty^2}\nabla\left(-\frac{\partial\phi}{\partial t} - \frac{1}{2}|\nabla\phi|^2\right)\right]$$

$$= a_\infty^2\nabla^2\phi\left[1 + \frac{\gamma-1}{a_\infty^2}\left(\frac{|q_\infty|^2}{2} - \frac{\partial\phi}{\partial t} - \frac{|\nabla\phi|^2}{2}\right)\right]$$

$$-\nabla\phi\cdot\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right)$$

Now

$$\nabla\phi\cdot\nabla\left(\frac{\partial\phi}{\partial t}\right) = \frac{1}{2}\frac{\partial}{\partial t}|\nabla\phi|^2$$

Hence,

$$\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}|\nabla\phi|^2 = a_\infty^2\nabla^2\phi\left[1 + \frac{\gamma-1}{a_\infty^2}\left(\frac{|q_\infty|^2}{2} - \frac{\partial\phi}{\partial t} - \frac{|\nabla\phi|^2}{2}\right)\right]$$

$$-\nabla\phi\cdot\nabla\left(\frac{1}{2}|\nabla\phi|^2\right) \tag{21}$$

where we can use the notation

$$\nabla\left(\tfrac{1}{2}|\nabla\phi|^2\right) = (\nabla\phi\cdot\nabla)\nabla\phi$$

Equation (21) is the full potential equation. Many authors, for example, Bendiksen,[5] simplify it by setting $\gamma = 1$.

TSD Equation

We proceed next to deduce from Eq. (21) the TSD equation. The derivation in Ref. 1 is marred by an arithmetic mistake, as we have noted, which the author goes on to compensate by ad hoc argument to obtain the correct form.

The small disturbance assumption is that the disturbance

$$\phi - \phi_\infty$$

is small, in the sense to be explained later. At the same time, we shall not need to make the assumption as in Ref. 1 that the angle of attack is zero, $\alpha = 0 = \beta$. Thus, we define

$$\phi = \phi_\infty + U\varphi \tag{22}$$

recalling that

$$|\boldsymbol{q}_\infty| = U$$

We shall write

$$\boldsymbol{q}_\infty = (i\boldsymbol{q}_1 + j\boldsymbol{q}_2 + k\boldsymbol{q}_3)U$$

where $\boldsymbol{q}_i$ are then direction cosines, and we would recover the zero-angle-of-attack case of Ref. 1 by setting

$$\boldsymbol{q}_2 = \boldsymbol{q}_3 = 0$$

What we do now is to substitute Eq. (22) into Eq. (21) making appropriate allowance for small φ. Thus, we have, with reference to Eq. (21),

$$\frac{\partial^2\phi}{\partial t^2} = U\frac{\partial^2\varphi}{\partial t^2}, \qquad \nabla^2\phi = U\nabla^2\varphi$$

$$|\nabla\phi|^2 = U^2 + 2Uq_\infty\cdot\nabla\varphi + U^2|\nabla\varphi|^2$$

$$= U^2 + \nabla\varphi\cdot\left[2Uq_\infty + U^2\nabla\varphi\right]$$

Hence,

$$\nabla\phi\cdot\nabla\left(\tfrac{1}{2}|\nabla\phi|^2\right) = (\boldsymbol{q}_\infty + U\nabla\varphi)\cdot\nabla\left(\tfrac{1}{2}U^2|\nabla\varphi|^2 + U\boldsymbol{q}_\infty\cdot\nabla\varphi\right)$$

Now

$$\nabla\left(\tfrac{1}{2}U^2|\nabla\varphi|^2\right) = iU^2\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2} + jU^2\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y^2} + kU^2\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial z^2}$$

$$+ \text{mixed partials}$$

where consistent with the arguments by Nixon[1] and others,[6–8] we may in our approximation omit the mixed partials. Next

$$\nabla\phi\cdot\nabla[Uq_\infty\cdot\nabla\varphi] = U^2\left(Uq_1 + U\frac{\partial\varphi}{\partial x}\right)\left(q_1\frac{\partial^2\varphi}{\partial x^2} + \text{mixed partials}\right)$$

$$+ U^2\left(Uq_2 + U\frac{\partial\varphi}{\partial y}\right)\left(q_2\frac{\partial^2\varphi}{\partial y^2} + \text{mixed partials}\right)$$

$$+ U^2\left(Uq_3 + U\frac{\partial\varphi}{\partial z}\right)\left(q_3\frac{\partial^2\varphi}{\partial z^2} + \text{mixed partials}\right)$$

Hence, omitting three-products,

$$(q_\infty + U\nabla\varphi)\cdot\nabla\left(\frac{1}{2}|U|^2|\nabla\varphi|^2 + Uq_\infty\cdot\nabla\varphi\right)$$

$$= Uq_1U^2\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2} + Uq_2U^2\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y^2} + Uq_3U^2\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial z^2}$$

$$+ U^2\left(Uq_1 + U\frac{\partial\varphi}{\partial x}\right)q_1\frac{\partial^2\varphi}{\partial x^2} + U^2\left(Uq_2 + U\frac{\partial\varphi}{\partial y}\right)q_2\frac{\partial^2\varphi}{\partial y^2}$$

$$+ U^2\left(Uq_3 + U\frac{\partial\varphi}{\partial z}\right)q_3\frac{\partial^2\varphi}{\partial z^2}$$

which on simplification yields

$$U^2Uq_1^2\frac{\partial^2\varphi}{\partial x^2} + U^2Uq_2^2\frac{\partial^2\varphi}{\partial y^2} + U^2Uq_3^2\frac{\partial^2\varphi}{\partial z^2}$$

$$+ 2\left(U^2Uq_1\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2} + U^2Uq_2\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y^2} + U^2Uq_3\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial z^2}\right)$$

Finally,

$$a_\infty^2\nabla^2\phi\left(\frac{\gamma-1}{a_\infty^2}\right)\left(\frac{|q_\infty|^2}{2} - \frac{\partial\phi}{\partial t} - \frac{|\nabla\phi|^2}{2}\right)$$

is approximated by

$$U(\gamma-1)\nabla^2\varphi\left(-U\frac{\partial\varphi}{\partial t} - Uq_\infty\cdot\nabla\varphi\right)$$

We omit the multiplication by $\partial\varphi/\partial t$, obtaining

$$-U^2(\gamma-1)\nabla^2\varphi\cdot q_\infty\cdot\nabla\varphi = -U^2(\gamma-1)\times\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}\right)\left(Uq_1\frac{\partial\varphi}{\partial x}+Uq_2\frac{\partial\varphi}{\partial y}+Uq_3\frac{\partial\varphi}{\partial z}\right)$$

where again we omit mixed derivatives and finally obtain

$$-U^2(\gamma-1)\left(Uq_1\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}+Uq_2\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y^2}+Uq_3\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial z^2}\right)$$

Also we approximate

$$\frac{\partial}{\partial t}|\nabla\phi|^2 \qquad \text{by} \qquad 2Uq_\infty\cdot\frac{\partial}{\partial t}\nabla\varphi$$

Putting it all together, we obtain, canceling the common factor U,

$$\begin{aligned}\frac{\partial^2\varphi}{\partial t^2}&+2U\left(q_1\frac{\partial^2\varphi}{\partial x\partial t}+q_2\frac{\partial^2\varphi}{\partial y\partial t}+q_3\frac{\partial^2\varphi}{\partial z\partial t}\right)\\&=a_\infty^2\left[1-M^2q_1^2-(1+\gamma)M^2q_1\frac{\partial\varphi}{\partial x}\right]\frac{\partial^2\varphi}{\partial x^2}\\&+a_\infty^2\left[1-M^2q_2^2-(1+\gamma)M^2q_2\frac{\partial\varphi}{\partial y}\right]\frac{\partial^2\varphi}{\partial y^2}\\&+a_\infty^2\left[1-M^2q_3^2-(1+\gamma)M^2q_3\frac{\partial\varphi}{\partial z}\right]\frac{\partial^2\varphi}{\partial z^2}\end{aligned}\tag{23}$$

which is then our generalized (for nonzero angle of attack) TSD equation, which of course reduces to the zero-angle-of-attack version when specialized to $\boldsymbol{q}_1=1$, $\boldsymbol{q}_2=\boldsymbol{q}_3=0$.

III. Solving the TSD Equation

Statement (23) of the TSD equation cannot be considered complete until we specify the boundary conditions. To keep the complexity of notation within bounds and at the same time reach toward a workable theory, we shall now consider only the case of a thin (zero thickness) rectangular wing of high aspect ratio so that we can specialize to typical section airfoil theory and two-dimensional aerodynamics in the X–Z plane, dispensing with the span or y variable, as, for example, in Ref. 5.

Thus, the TSD becomes

$$\begin{aligned}\frac{\partial^2\varphi}{\partial t^2}&+2U\left(\cos\alpha\frac{\partial^2\varphi}{\partial x\partial t}+\sin\alpha\frac{\partial^2\varphi}{\partial z\partial t}\right)\\&=a_\infty^2\left[(1-M^2\cos^2\alpha)-M^2(1+\gamma)\cos\alpha\frac{\partial\varphi}{\partial x}\right]\frac{\partial^2\varphi}{\partial x^2}\\&+a_\infty^2\left[(1-M^2\sin^2\alpha)-M^2(1+\gamma)\sin\alpha\frac{\partial\varphi}{\partial z}\right]\frac{\partial^2\varphi}{\partial z^2}\end{aligned}\tag{24}$$

Our first boundary condition is the attached-flow condition

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0+}=\left.\frac{\partial\phi_\infty}{\partial z}\right|_{z=0}+w_a(x,t), \qquad |x|<b$$

or

$$U\left.\frac{\partial\varphi}{\partial z}\right|_{z=0}=w_a(x,t), \qquad |x|<b \tag{25}$$

where $w_a(x,t)$ is the downwash and b is the half-chord. In terms of φ, Eq. (25) becomes

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0+}=w_a(x,t), \qquad |x|<b \tag{25a}$$

To describe the Kutta–Joukowsky conditions, we need first to define the acceleration potential. The acceleration

$$\frac{\mathrm{D}q}{\mathrm{D}t}=\frac{\partial q}{\partial t}+(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}\tag{26}$$

$$\nabla\left(\frac{1}{2}|\boldsymbol{q}|^2\right)=(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}$$

Hence,

$$\frac{\mathrm{D}q}{\mathrm{D}t}=\nabla\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2\right)$$

or the acceleration potential ψ is given by

$$\psi=\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2, \qquad \psi_\infty=\frac{1}{2}|\boldsymbol{q}_\infty|^2=\frac{1}{2}U^2$$

so that

$$\psi=U\frac{\partial\varphi}{\partial t}+\frac{1}{2}|\boldsymbol{q}_\infty+U\nabla\varphi|^2$$

which, consistent with the small disturbance assumption, we may approximate by

$$\begin{aligned}\psi&=U\frac{\partial\varphi}{\partial t}+\frac{1}{2}U^2+U(q_\infty\cdot\nabla\varphi)\\&=\frac{1}{2}U^2+U\frac{\partial\varphi}{\partial t}+U^2\left[\cos\alpha\frac{\partial\varphi}{\partial x}+\sin\alpha\frac{\partial\varphi}{\partial z}\right]\\&=\psi_\infty+U\frac{\partial\varphi}{\partial t}+U^2\left[\cos\alpha\frac{\partial\varphi}{\partial x}+\sin\alpha\frac{\partial\varphi}{\partial z}\right]\end{aligned}$$

We need to consider the pressure p next. As we have seen in Sec. II,

$$p=\left(a_\infty^2\big/\gamma\rho_\infty^{\gamma-1}\right)\rho^\gamma=\left(a_\infty^2\rho_\infty\big/\gamma\right)(\rho/\rho_\infty)^\gamma$$

which by Eq. (18) is equal to

$$\frac{a_\infty^2\rho_\infty}{\gamma}\left[1+\frac{\gamma-1}{a_\infty^2}(\psi_\infty-\psi)\right]^{\gamma/(\gamma-1)}\tag{27}$$

and

$$\frac{\gamma-1}{a_\infty^2}(\psi_\infty-\psi)=(\gamma-1)M^2\left(\cos\alpha\frac{\partial\varphi}{\partial x}+\sin\alpha\frac{\partial\varphi}{\partial z}\right)+\frac{(\gamma-1)M}{a_\infty}\frac{\partial\varphi}{\partial t}$$

which is less than one in magnitude. As in Refs. 1 and 3, we may approximate expression (27) by

$$\left(a_\infty^2\rho_\infty\big/\gamma\right)\left[1+\left(\gamma\big/a_\infty^2\right)(\psi_\infty-\psi)\right]\tag{28}$$

Let

$$\tilde\psi=\psi-\psi_\infty=U\left(\frac{\partial\varphi}{\partial t}+U\cos\alpha\frac{\partial\varphi}{\partial x}+U\sin\alpha\frac{\partial\varphi}{\partial z}\right)\tag{28a}$$

Then

$$\delta p=p|_{z=0+}-p|_{z=0-}=-\rho_\infty\delta\tilde\psi\tag{29}$$

$$\delta\tilde\psi=\tilde\psi|_{z=0+}-\tilde\psi|_{z=0-}$$

Hence, the Kutta–Joukowsky condition, zero pressure jump off the chord and at the trailing edge, becomes

$$\delta\tilde{\psi}\left\{\begin{aligned} &= 0, \quad |x| > b \\ &= 0, \quad x \to b-, \quad 0 \le |\alpha| < \pi/2 \end{aligned}\right. \tag{30}$$

Linear TSD Equation: Generalized Possio Integral Equation

To proceed to the solution of Eq. (24) subject to the boundary conditions (25) and (30), we need first to introduce the linear (or linearized, as we shall show later) TSD, which is obtained by setting the quasi-linear (product) term in Eq. (24) to be zero. Thus, we have the linear TSD or subsonic SD equation given by

$$\frac{\partial^2\varphi}{\partial t^2} + 2U\left(\cos\alpha\frac{\partial^2\varphi}{\partial x\partial t} + \sin\alpha\frac{\partial^2\varphi}{\partial z\partial t}\right) = a_\infty^2(1 - M^2\cos^2\alpha)\frac{\partial^2\varphi}{\partial x^2} + a_\infty^2(1 - M^2\sin^2\alpha)\frac{\partial^2\varphi}{\partial z^2}$$

$$-\infty < x, z, \infty, \qquad 0 \le t \tag{24L}$$

subject of course to the same boundary conditions (25) and (30), all expressible in terms of the disturbance potential φ.

We may proceed exactly as in Ref. 9, where we consider the case $\alpha = 0$, but with fewer technical mathematical details than therein. Thus, we take Laplace transforms in the time variable t and Fourier transforms in the space variable x,

$$\hat{\varphi}(x, z, \lambda) = \int_0^\infty e^{-\lambda t}\varphi(x, z, t)\,\mathrm{d}t, \qquad Re\lambda > \sigma_a$$

$$\hat{\hat{\varphi}}(i\omega, z, \lambda) = \int_{-\infty}^\infty e^{-i\omega x}\hat{\varphi}(x, z, \lambda)\,\mathrm{d}x, \qquad -\infty < \omega < \infty$$

Then Eq. (24L) yields

$$a_\infty^2(1 - M^2\sin^2\alpha)\frac{\partial^2\hat{\hat{\varphi}}}{\partial z^2} - (2\lambda U\sin\alpha + 2U^2 i\omega\sin\alpha\cos\alpha)\frac{\partial\hat{\hat{\varphi}}}{\partial z} - \left(\lambda^2 + \omega^2 a_\infty^2(1 - M^2\cos^2\alpha) + 2\lambda U i\omega\cos\alpha\right)\hat{\hat{\varphi}} = 0 \tag{31}$$

Let, in the usual notation, k denote the reduced frequency,

$$k = \lambda b/U \tag{32}$$

so that in particular in what follows we may take $b = 1$. Let

$$a_1 = (1 - M^2\sin^2\alpha)$$

$$a_2 = M^2(k + i\omega\cos\alpha)\sin\alpha$$

$$a_3 = k^2M^2 + 2kM^2 i\omega\cos\alpha + \omega^2(1 - M^2\cos^2\alpha)$$

Then Eq. (31) can be expressed

$$a_1\frac{\mathrm{d}^2\hat{\hat{\varphi}}}{\mathrm{d}z^2} - 2a_2\frac{\mathrm{d}\hat{\hat{\varphi}}}{\mathrm{d}z} - a_3\hat{\hat{\varphi}} = 0$$

Hence, to satisfy the far-field conditions,

$$\begin{aligned}\hat{\hat{\varphi}}(i\omega, z, \lambda) &= A_+e^{r_1z}, \qquad z > 0 \\ &= A_-e^{r_2z}, \qquad z < 0\end{aligned}$$

where

$$r_1 = \left(a_2 - \sqrt{a_2^2 + a_1a_3}\right)\Big/a_1, \qquad Re r_1 < 0$$

$$r_2 = \left(a_2 + \sqrt{a_2^2 + a_1a_3}\right)\Big/a_1, \qquad Re r_2 > 0$$

Thus defined, we note that $\hat{\hat{\varphi}}$ is not continuous at $z = 0$. Let

$$\hat{\hat{\nu}} = \left.\frac{\partial\hat{\hat{\varphi}}}{\partial z}\right|_{z=0}$$

Then

$$\hat{\hat{\varphi}}(i\omega, z, \lambda) = (\hat{\hat{\nu}}/r_1)e^{r_1z}, \qquad z > 0 \tag{33}$$

$$\hat{\hat{\varphi}}(i\omega, z, \lambda) = (\hat{\hat{\nu}}/r_2)e^{r_2z}, \qquad z < 0 \tag{34}$$

where we should note that $\hat{\hat{\nu}}$ is continuous at $z = 0$. Next we define the Laplace transform of the acceleration potential:

$$\hat{\psi}(x, z, \lambda) = \int_0^\infty e^{-\lambda t}\tilde{\psi}(x, z, t)\,\mathrm{d}t$$

where

$$\tilde{\psi}(x, z, t) = U\left[\frac{\partial\varphi}{\partial t} + U\cos\alpha\frac{\partial\varphi}{\partial x} + U\sin\alpha\frac{\partial\varphi}{\partial z}\right]$$

and define

$$\begin{aligned}\hat{\hat{\psi}}(i\omega, z, \lambda) &= \int_{-\infty}^\infty e^{-i\omega x}\hat{\psi}(x, z, \lambda)\,\mathrm{d}x \\ &= U\left[(\lambda + Ui\omega\cos\alpha)\hat{\hat{\varphi}}(i\omega, z, \lambda) + U\sin\alpha\frac{\partial\hat{\hat{\phi}}}{\partial z}(i\omega, z, \lambda)\right]\end{aligned} \tag{35}$$

In particular it follows that, because $\partial\hat{\hat{\phi}}/\partial z$ is continuous at $z = 0$,

$$\delta\hat{\hat{\psi}} = U(\lambda + Ui\omega\cos\alpha)[\hat{\hat{\varphi}}(i\omega, 0+, \lambda) - \hat{\hat{\varphi}}(i\omega, 0-, \lambda)]$$

which by Eqs. (33) and (34) is equal to

$$U[\lambda + Ui\omega\cos\alpha](1/r_1 - 1/r_2)\hat{\hat{\nu}} \tag{36}$$

As is customary, we now define the Küssner doublet function:

$$\begin{aligned}A(x, t) &= \delta p/\rho_\infty U \\ &= -\delta\tilde{\psi}/U, \qquad |x| < 1 \\ &= 0, \qquad |x| > 1 \\ &= 0, \qquad x = 1-\end{aligned}$$

Let

$$\hat{A}(x, \lambda) = \int_0^\infty A(x, t)e^{-\lambda t}\,\mathrm{d}t, \qquad Re\lambda > \sigma_a$$

$$\hat{\hat{A}}(i\omega, \lambda) = \int_{-1}^1 e^{-i\omega x}\hat{A}(x, \lambda)\,\mathrm{d}x$$

Then, we obtain from Eq. (36) that

$$\hat{\hat{A}}(i\omega, \lambda) = (\lambda + Ui\omega\cos\alpha)\hat{\hat{\nu}}(i\omega, \lambda)(1/r_1 - 1/r_2)$$

$$-\infty < \omega < \infty$$

or

$$\hat{\hat{\nu}}(i\omega, \lambda) = \{[1/(\lambda + i\omega U\cos\alpha)][1/(1/r_1 - 1/r_2)]\}\hat{\hat{A}}(i\omega, \lambda) \tag{37}$$

which checks with the usual formula for $\alpha = 0$, as given in Ref. 9. This is then the generalization of the Possio integral equation for nonzero angle of attack in the spatial frequency transform version. Taking inverse Fourier transforms, let

$$\int_{-\infty}^{\infty} \hat{P}(x,\lambda)e^{-i\omega x} = \left(\frac{1}{k+i\omega\cos\alpha}\right)\frac{1}{(1/r_1 - 1/r_2)} \qquad -\infty < \omega < \infty \tag{38}$$

Then Eq. (37) takes the more familiar form

$$\hat{w}_a(x,\lambda) = \int_{-1}^{1} \hat{P}(x-\xi,\lambda)\hat{A}(\xi,\lambda)\,\mathrm{d}\xi, \qquad |x| < 1 \tag{39}$$

Here the multiplier (38)

$$[1/(k+i\omega\cos\alpha)][1/(1/r_1 - 1/r_2)]$$

can be expressed as

$$\frac{1}{2}\left(\frac{1}{k+i\omega\cos\alpha}\right)\left(\frac{M^2k^2 + 2M^2ki\omega\cos\alpha + \omega^2(1-M^2\cos^2\alpha)}{\sqrt{M^2k^2+2M^2ki\omega\cos\alpha+\omega^2(1-M^2)}}\right) \tag{40}$$

because by Eq. (25)

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0} = \frac{1}{U}w_a(x,t), \qquad |x| < 1$$

We assume now that Eq. (39), as in the familiar case for $\alpha = 0$, has a unique solution with

$$\hat{A}(x,\lambda) = 0, \qquad x = 1-$$

Linear Nonhomogeneous TSD Equation

In developing the solution to the nonlinear TSD equation (24), we need to continue with the linear equation (24L) but now the nonhomogeneous case, nonzero right-hand side. Thus, we need to consider

$$\frac{\partial^2\varphi}{\partial t^2} + 2U\left(\cos\alpha\frac{\partial^2\varphi}{\partial x\partial t} + \sin\alpha\frac{\partial^2\varphi}{\partial z\partial t}\right) - a_\infty^2(1-M^2\cos^2\alpha)\frac{\partial^2\varphi}{\partial x^2} - a_\infty^2(1-M^2\sin^2\alpha)\frac{\partial^2\varphi}{\partial z^2} = f(x,z,t), \qquad -\infty < x, z < \infty, \qquad 0 < t \tag{24NL}$$

with zero initial conditions,

$$\varphi(x,z,0) = 0, \qquad \dot{\varphi}(x,z,0) = 0$$

and zero boundary conditions,

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0} = 0, \qquad |x| < 1$$

$$\delta\tilde{\psi} = 0, \qquad -\infty < x < \infty$$

and zero far-field conditions. In that case we can show that Eq. (24NL) has a unique solution given in terms of a Green's function,

$$\varphi(x,z,t) = \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-\xi, y-\eta, t-\sigma)f(\xi,\eta,\sigma)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma \qquad -\infty < x, z < \infty \tag{41}$$

subject of course to some constraints on the forcing function $f(\cdot,\cdot,\cdot)$, which are too technical to go into as is also the proof. It will be convenient notationally to denote the solution as a linear transformation (operator) $\mathcal{L}$:

$$\varphi = \mathcal{L}f \tag{42}$$

Note here, another mathematical technicality, that the manner in which φ in Eq. (41) satisfies Eq. (24NL) may depend on the differentiability conditions imposed on f, and in particular, second-order derivatives of φ may not be defined at every point. In other words, shocks can occur.

We turn finally now to the solution of the nonlinear TSD equation (24) with the specified boundary conditions. In Ref. 1 Nixon assumes that the solution can be expanded "as a perturbation series

$$\varphi(x,z,t) = \phi_0(x,z) + \varepsilon\phi_1(x,z,t)\cdots$$

[Eq. (38) in Ref. 1] where ε is some small parameter." Our approach here is totally different, in spite of a superficial similarity.

Thus, we consider the solution of Eq. (24), which (the initial conditions being zero) depends only on the specified downwash function $w_a(\cdot,t)$ where φ is the output corresponding to the input $w_a(\cdot,t)$, and our assumption that the output is an analytic function of the input. In other words, let λ be a complex variable

$$\varphi(\lambda)$$

denoting the output corresponding to

$$\lambda w_a(\cdot,t)$$

Then $\varphi(\lambda)$ can be expanded in a power series in λ about $\lambda = 0$, valid for any λ in the finite part of the plane:

$$\varphi(\lambda) = \sum_0^\infty \lambda^k\varphi_k \tag{43}$$

where

$$\phi(0) = 0$$

$$\varphi_k = \left.\frac{(\mathrm{d}^k/\mathrm{d}\lambda^k)\varphi(\lambda)}{k!}\right|_{\lambda=0} \tag{44}$$

Now by Eq. (24), where $\varphi(\lambda)$ is more explicitly

$$\varphi(x,z,t;\lambda)$$

we have

$$\frac{\partial^2\varphi(\lambda)}{\partial t^2} + 2U\left(\cos\alpha\frac{\partial^2\varphi(\lambda)}{\partial x\partial t} + \sin\alpha\frac{\partial^2\varphi(\lambda)}{\partial z\partial t}\right) - a_\infty^2(1-M^2\cos^2\alpha)\frac{\partial^2\varphi(\lambda)}{\partial x^2} - a_\infty^2(1-M^2\sin^2\alpha)\frac{\partial^2\varphi(\lambda)}{\partial z^2} = U^2(1+\gamma)\cos\alpha\frac{\partial\varphi(\lambda)}{\partial x}\frac{\partial^2\varphi(\lambda)}{\partial x^2} + U^2(1+\gamma)\sin\alpha\frac{\partial\varphi(\lambda)}{\partial z}\frac{\partial^2\varphi(\lambda)}{\partial z^2} \tag{45}$$

$$U\left.\frac{\partial\varphi(\lambda)}{\partial z}\right|_{z=0} = \lambda w_a(t,x), \qquad |x| < 1 \tag{46}$$

Hence, differentiating with respect to λ in Eq. (46),

$$U\left.\frac{\partial}{\partial z}\frac{\partial\varphi(\lambda)}{\partial\lambda}\right|_{z=0} = w_a(t,x), \qquad |x| < 1 \tag{47}$$

$$U\left.\frac{\partial}{\partial z}\frac{\partial^k\varphi(\lambda)}{\partial\lambda^k}\right|_{z=0} = 0, \qquad |x| < 1, \qquad k \geq 2 \tag{48}$$

and in Eq. (45),

$$\frac{\partial^2\varphi_k}{\partial t^2}+2U\left(\cos\alpha\frac{\partial^2\varphi_k}{\partial x\partial t}+\sin\alpha\frac{\partial^2\varphi_k}{\partial z\partial t}\right)$$
$$-a_\infty^2(1-M^2\cos^2\alpha)\frac{\partial^2\varphi_k}{\partial x^2}-a_\infty^2(1-M^2\sin^2\alpha)\frac{\partial^2\varphi_k}{\partial z^2}$$
$$=k!U^2(1+\gamma)\cos\alpha\frac{\partial^k}{\partial\lambda^k}\left[\frac{\partial\varphi(\lambda)}{\partial x}\frac{\partial^2\varphi(\lambda)}{\partial x^2}\right]\Bigg|_{\lambda=0}$$
$$+k!U^2(1+\gamma)\sin\alpha\frac{\partial^k}{\partial\lambda^k}\left[\frac{\partial\varphi(\lambda)}{\partial z}\frac{\partial^2\varphi(\lambda)}{\partial z^2}\right]\Bigg|_{\lambda=0} \tag{49}$$

Hence, for $k=1$ we see that Eq. (49) reduces to the linear TSD equation (24L) with the associated boundary conditions. The solution φ_1 is then uniquely determined via the corresponding nonzero-angle-of-attack Possio equation (39).

Next we see that φ_2 satisfies

$$\frac{\partial^2\varphi_2}{\partial t^2}+2U\left(\cos\alpha\frac{\partial^2\varphi_2}{\partial x\partial t}+\sin\alpha\frac{\partial^2\varphi_2}{\partial z\partial t}\right)$$
$$-a_\infty^2(1-M^2\cos^2\alpha)\frac{\partial^2\varphi_2}{\partial x^2}-a_\infty^2(1-M^2\sin^2\alpha)\frac{\partial^2\varphi_2}{\partial z^2}$$
$$=2U^2(1+\gamma)\cos\alpha\left(2\frac{\partial\varphi_1}{\partial x}\frac{\partial^2\varphi_1}{\partial x^2}\right)$$
$$+2U^2(1+\gamma)\sin\alpha\left(2\frac{\partial\varphi_1}{\partial z}\frac{\partial^2\varphi_1}{\partial z^2}\right) \tag{50}$$

with

$$\frac{\partial\varphi_1}{\partial z}\bigg|_{z=0}=0, \qquad |x|<1 \tag{51}$$

However, this is the linear nonhomogeneous equation (24NL) we have already treated. With f_2 denoting the right-hand side of Eq. (50), we have that

$$\varphi_2=\mathcal{L}f_2$$

More generally, with

$$f_k=k!U^2(1+\gamma)\cos\alpha\frac{\partial^k}{\partial\lambda^k}\left[\frac{\partial\varphi(\lambda)}{\partial x}\frac{\partial^2\varphi(\lambda)}{\partial x^2}\right]\Bigg|_{\lambda=0}$$
$$+k!U^2(1+\gamma)\sin\alpha\frac{\partial^k}{\partial\lambda^k}\left[\frac{\partial\varphi(\lambda)}{\partial z}\frac{\partial^2\varphi(\lambda)}{\partial z^2}\right]\Bigg|_{\lambda=0}$$

we have that

$$\varphi_k=\mathcal{L}f_k, \qquad k\geq 2$$

Hence, we obtain

$$\varphi(\lambda)=\lambda\varphi_1+\sum_2^\infty\lambda^k\mathcal{L}f_k \tag{52}$$

or taking $\lambda=1$, the solution to Eq. (24) is given by

$$\varphi=\varphi_1+\sum_2^\infty\mathcal{L}f_k \tag{53}$$

Our main interest in Eq. (53) is that of stability.

Theorem 3. Suppose the linear solution φ_1 is stable. That is, denote the dependence on t by $\varphi_1(\cdot,t)$ and suppose

$$\varphi_1(\cdot,t)\to 0 \qquad \text{as} \qquad t\to\infty$$

then so does $\varphi(\cdot,t)$. Also suppose $\varphi_1(\cdot,t)$ is periodic in t, then so is $\varphi(\cdot,t)$ with the same period.

Remark. We note that these statements are consistent with the central result of the more general Hopf bifurcation theory, as treated, for example, in Ref. 3. In particular, stability or instability is determined by the linearized equation, a result of considerable importance to the determination of flutter speed.

Proof. These results are easily deduced from the expansion (53). Thus, if φ_1 is stable, so is $\mathcal{L}(f_k)$ for each k. Similarly if φ_1 is periodic, so is each f_k and then also $\mathcal{L}(f_k)$ with the same period. The technical mathematical details are too long to be included here.

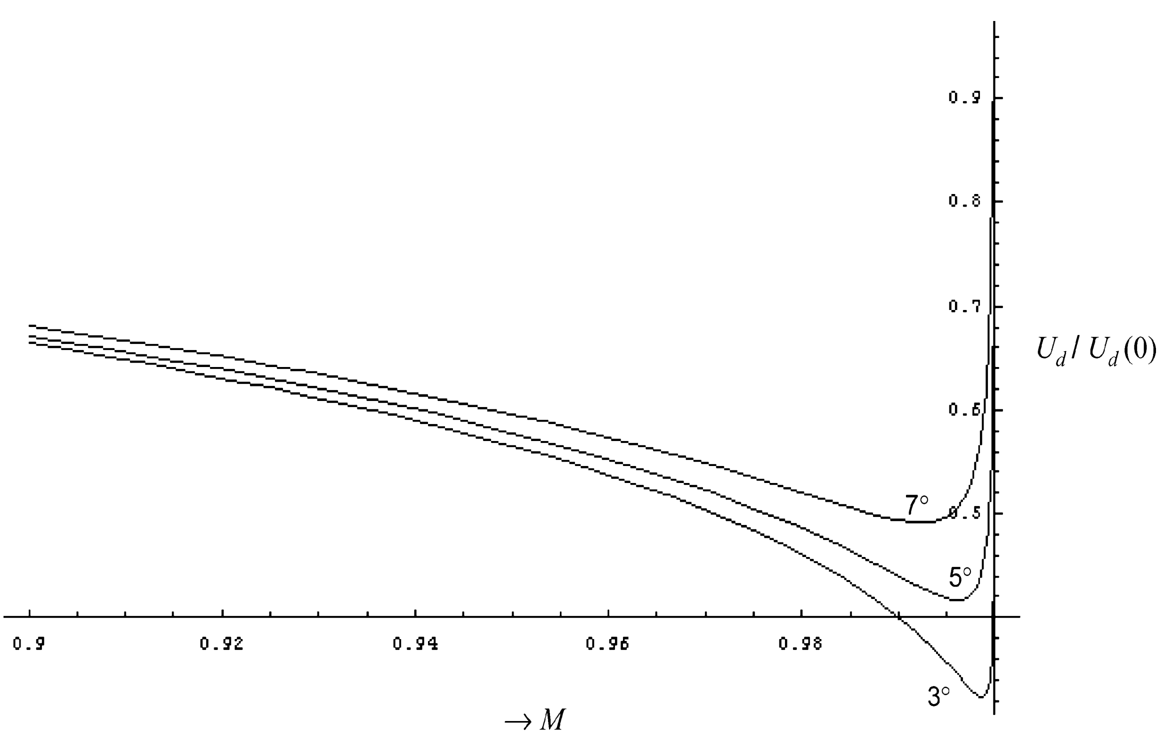

Fig. 1 Dip in divergence speed due to nonzero angle of attack, $U_d(0)$: divergence speed at M = 0 and α = 0.

IV. Discussion

We have presented a new solution of the TSD equation for an airfoil of zero thickness, generalized to include nonzero angle of attack, from which we deduce results on stability consistent with the more general Hopf bifurcation theory. The fact that stability is determined by the linearized equation is of considerable importance in determining flutter speed, in particular for theoretic verification of the dependence on the angle of attack. Here we shall consider the other important stability index, the divergence speed, specializing to a slender high aspect ratio straight wing, where the downwash function $w_a(x, t)$ is given by

$$w_a(x, t) = -\dot{h}(t) - (x - a)\dot{\theta}(t) - U \cos\alpha\theta(t) \qquad (54)$$

where h is the plunge displacement and θ is the pitch angle. Then following the theory developed in Ref. 10 for the case of zero angle of attack, we can readily extend the formula therein for the divergence speed to

$$U_d = \frac{\sqrt{\pi}}{2} \frac{\sqrt{GJ}}{\ell b\sqrt{\rho(1 + 2a)(\cos^2\alpha)}} \left(\frac{1 - M^2\cos^2\alpha}{\sqrt{1 - M^2}}\right)^{\frac{1}{2}} \qquad 0 \le M \le 1 \quad (55)$$

Note that for $\alpha \neq 0$, U_d becomes infinite at $M = 1$. We have a transonic dip at

$$M = \sqrt{1 - \tan^2\alpha} \qquad |\alpha| \le \pi/4$$

Figure 1 shows U_d vs M for $\alpha = 3$, 5, and 7 deg, illustrating the dip phenomenon. Note that the dip occurs at smaller value of M as α increases. This dependence is similar to that reported in experimental/computational work in Refs. 11 and 12 for the flutter speed.

Although the extension of this theory to the case of nonzero thickness would be of interest, note that we have shown the existence of a transonic dip even for zero thickness due to nonzero angle of attack.

V. Conclusions

An analytical solution is presented to the transonic small disturbance potential equation with the Kutta–Joukowsky boundary conditions for an airfoil of zero thickness, generalizing it at the same time to nonzero angle of attack. In particular, an extension of the Possio integral equation is derived that is valid for nonzero angle of attack, as well as a formula for the divergence speed, showing explicitly a transonic dip, depending on the angle of attack.

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